Improvements on Circuit Lattices

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Introduction. Outline

- Motivation
- Definitions
- Basic Algorithms

Motivation

- One way function $x \to f(x)$
- E.g. $x \to a^x \mod p$
- M - plain-text,
  K - key,
  $E_K(M)$ cipher-text in DES(AES):

$$K \to E_K(M)$$

- Still one-way

Motivation

- **Compute** with low number of small gates

$$f(x_1, x_2, x_3, x_4) = F(g_1(x_1, x_2), g_2(x_2, x_3), g_3(x_3, x_4))$$

- **Invert:** solve $f(x) = y$ in x
- **Simplify:** introduce new variables

$$f(x_1, x_2, x_3, x_4) = y \Leftrightarrow \begin{array}{r} g_1(x_1, x_2) = y_1 \\ g_2(x_2, x_3) = y_2 \\ g_3(x_3, x_4) = y_3 \\ F(y_1, y_2, y_3) = y \end{array}$$

- 3-sparse equations system

DES and TripleDES equations

- 64-bit plain-text, cipher-text, convenient to write variables
- 64-bit internal state blocks and 56(112)-bit key
- Equations from S-boxes(6-bit$\rightarrow$ 4-bit)

$$Y_4 \oplus Z_4 = S(X_6 \oplus K_6)$$

- 20 variables(20-sparse), 2^{16} solutions each
- DES: 632 variables, 128 equations
- TDES: 1712 variables, 384 equations

Zakrevskij-Raddum representation

- $f_i(X_i) = 0 \Leftrightarrow$ solutions V_i in variables $X_i \Leftrightarrow E_i = (X_i, V_i)$

$$x_1 x_2 + x_3 \equiv 0 \mod 2 \Leftrightarrow \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}$$

- **Solve with**:
- Gluing(enlarge equations by combining)
- Guess variable values
- Pairwise Agreeing(propagation, decision)

Local Reduction (Pairwise Agreeing)

x_1	x_2	x_3
0	0	1
0	0	0
0	**1**	**0**
1	1	1

x_1	x_2	x_4
0	0	0
1	**0**	**1**
1	1	0
1	1	1

- Common variables $\{x_1, x_2\}$
- Projections on $\{x_1, x_2\}$:
- $00, 01, 11$ and $00, 10, 11$
- Remove vectors with projection not in the projections of another list
- Due to [Zakrevskij-Vasilkova,00] and [Raddum,04]

Agreeing Algorithm

- **Repeat**:
- Find E_i and E_j which disagree
- Remove some local solutions in E_i or E_j and make them agree.

Related Algorithms Running Time(q=2)

n l-sparse Boolean equations in n variables

$l =$	3	4	5	6
the worst case	1.324^n	1.474^n	1.569^n	1.637^n
expectation[Semaev,10]	1.029^n	1.107^n	1.182^n	1.239^n

.

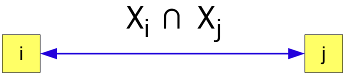
- Worst and average cases of the problem are excitingly different
- A software implementation is comparable with SAT-solvers in speed[Schilling-Raddum,10]

Circuit Lattices. Contribution Outline

- Equation Graph simplification, New versus [Semaev,WCC'09]
- A faster agreeing [Raddum-Semaev,07]
- Circuit Lattices, New versus [WCC'09]
- Circuit Lattice for TripleDES

Equation Graph and Pairwise Agreeing

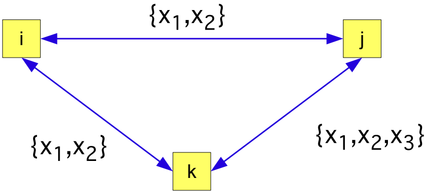
- Connect $E_i = (X_i, V_i)$ and $E_j = (X_j, V_j)$ by



- if $X_i \cap X_j \neq \emptyset$
- **Pairwise Agreeing:**
- Learn $X_i \cap X_j \neq a$ from E_i. Expand to E_j
- or vice versa

Obsolete Edges

- Remove some edges and keep Algorithm's output



- 6 connections(arcs) initially

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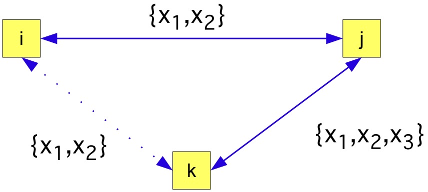

- 4 connections(arcs) as in WCC'09

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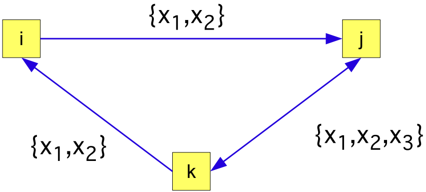

- 4 connections(arcs) now

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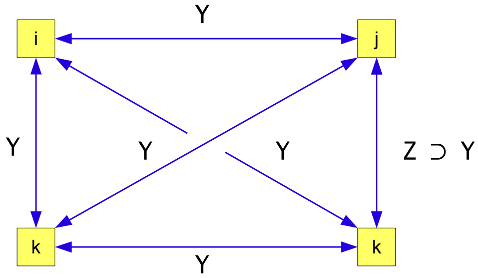

- 12 connections(arcs) initially

Obsolete Edges

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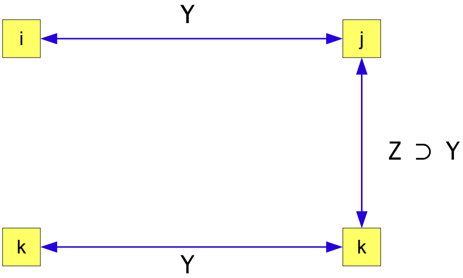

- 6 connections(arcs) as in WCC'09

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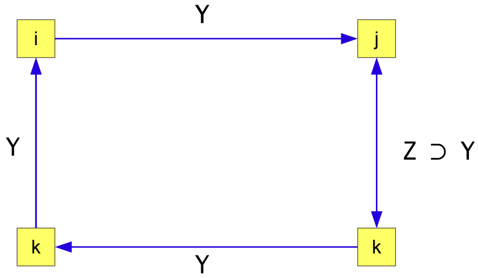

- 5 connections(arcs) now

Hidden Cyclic Structure

$$\{x_1, x_2, x_4, x_6\}, \{x_2, x_3, x_5, x_7\}, \{x_1, x_3, x_4, x_6\}, \{x_2, x_4, x_5, x_7\},$$
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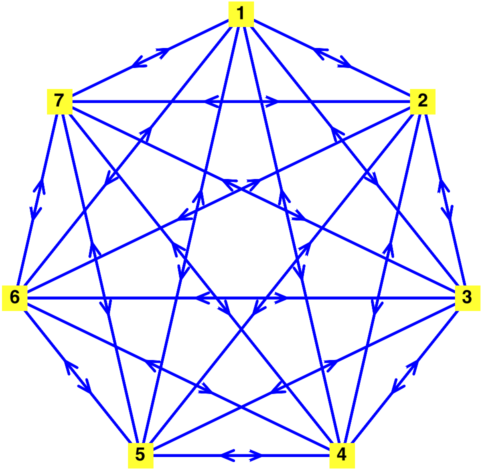

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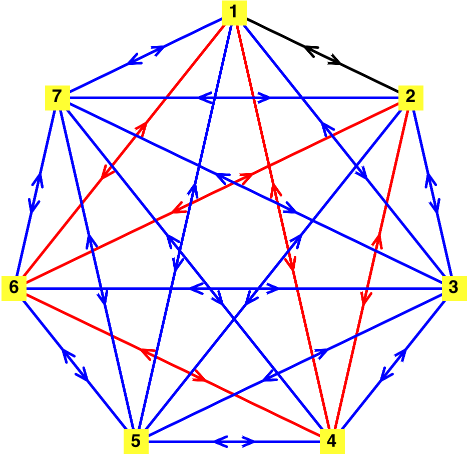

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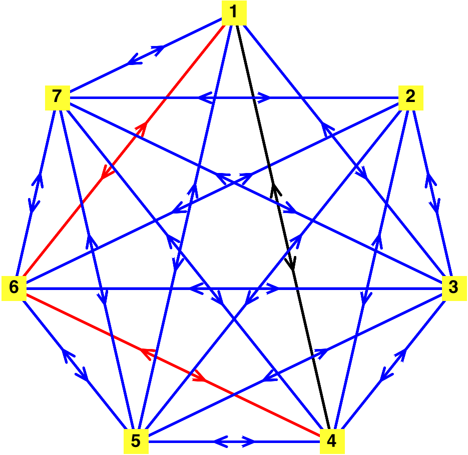

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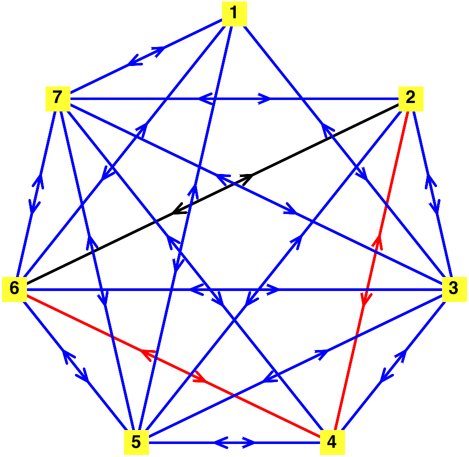

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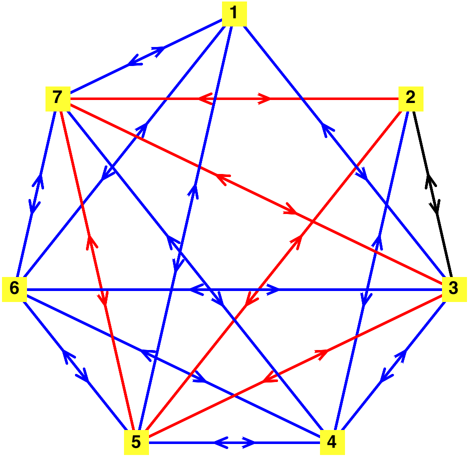

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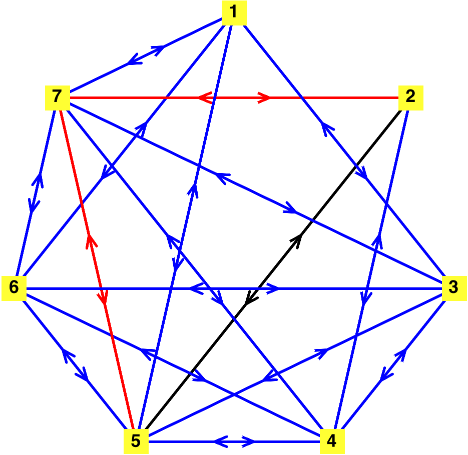

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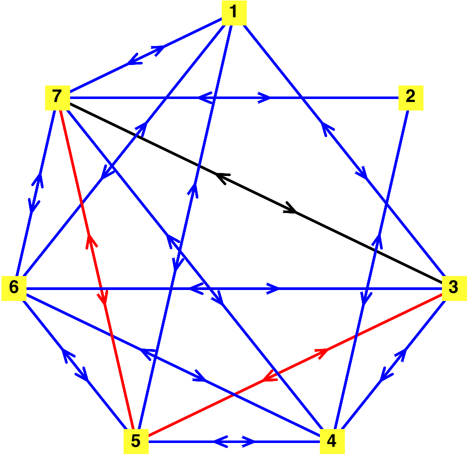

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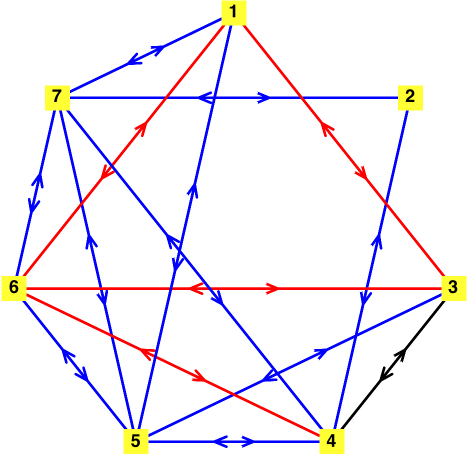

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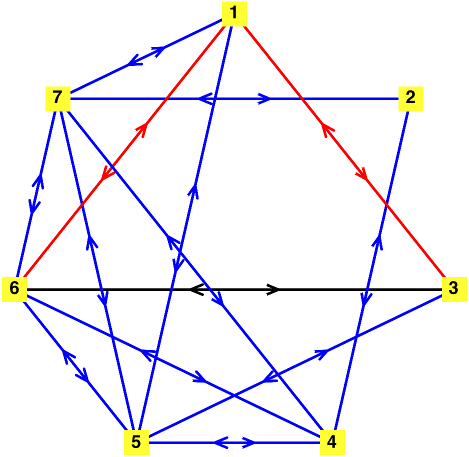

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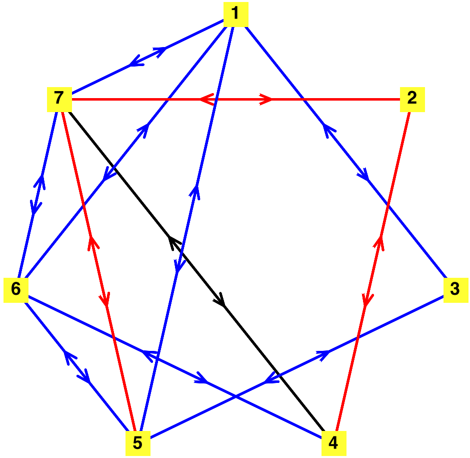

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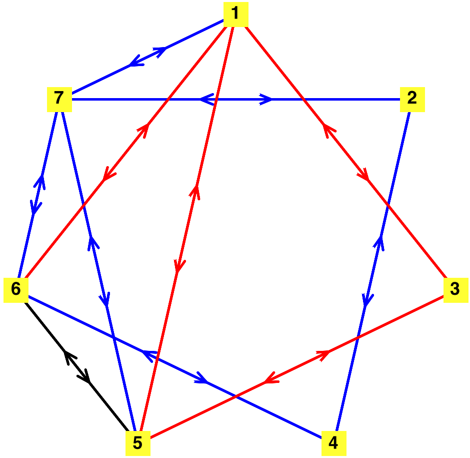

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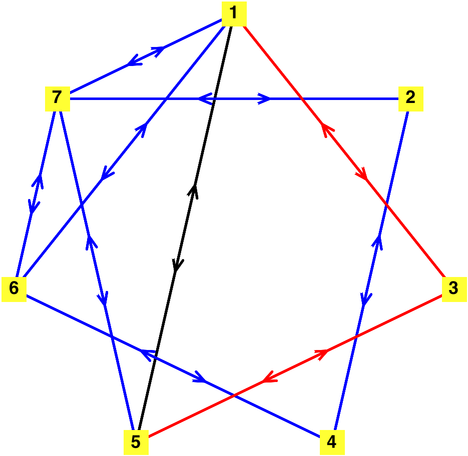

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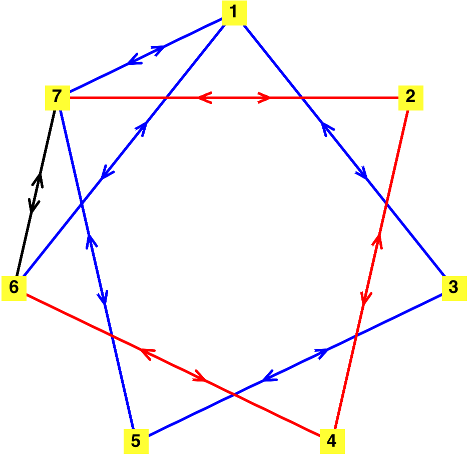

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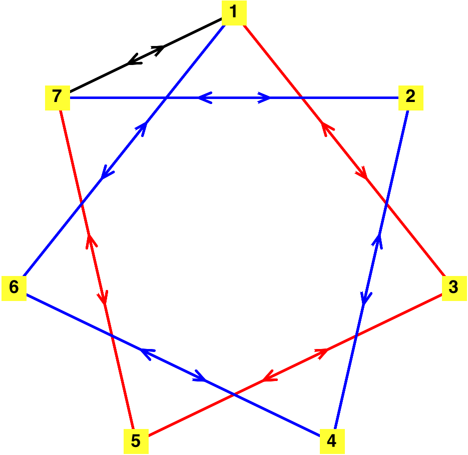

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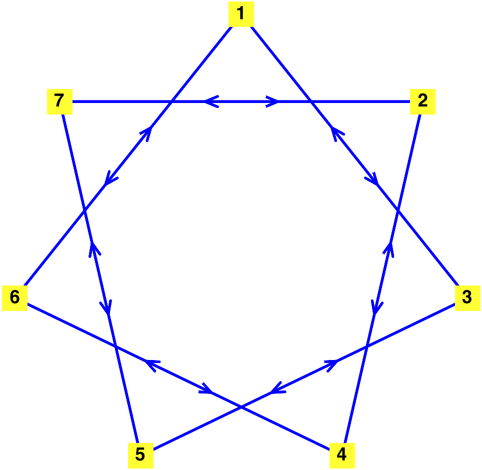
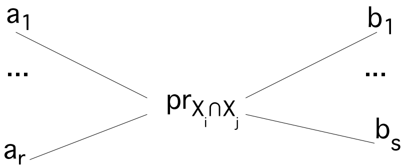


Faster Pairwise Agreeing

- $E_i \to E_j$
- $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ local solutions to E_i and E_j
- with the same projection to $X_i \cap X_j$



- Pre-compute all such tuples $(a_1, \ldots, a_r; b_1, \ldots, b_s)$

Faster Pairwise Agreeing

- **Notation:** $a_i \neq$ part of a global solution $\Rightarrow$ mark $\bar{a}_i$
- $(a_1, \ldots, a_r; b_1, \ldots, b_s)$ equivalent to
- $\bar{a}_1, \ldots, \bar{a}_r \Rightarrow \bar{b}_1, \ldots, \bar{b}_s$
- **Solving the system**:
- Introduce a guess $\equiv$ mark some of a_i
- Expand marking through the imlications

Example

- Equations by local solutions:

	x_1	x_2	x_3
a_1	0	0	1
a_2	0	1	1
a_3	1	1	0

,

	x_1	x_4
b_1	0	1
b_2	1	0

,

	x_2	x_3	x_4
c_1	0	1	1
c_2	1	0	1
c_3	1	1	0

- Tuples

$$
\begin{array}{lll}
(a_1, a_2; b_1), (b_1; a_1, a_2), & (a_3, b_2), (b_2; a_3), & (b_1; c_2), (c_2; b_1) \\
(c_1, c_3; b_2), (b_2; c_1, c_3), & (a_1; c_1), (c_1; a_1), & \\
(a_2; c_3),\ (c_2; a_3) & &
\end{array}
$$

Example

- Assume $x_4 = 0 \Rightarrow b_1$ should be marked(wrong local solution)
- Marking expansion

$$\begin{tikzcd}
(b_1;\, a_1, a_2) \arrow[r] \arrow[d] & (a_1;\, c_1) \arrow[d] & & & \\
(a_2;\, c_3) \arrow[r] & (c_1, c_3;\, b_2) \arrow[r] & (b_2;\, a_3) \arrow[r] & (a_3;\, c_2) \arrow[r] & (c_2;\, b_1)
\end{tikzcd}$$

- All instances(b_2 at early stage) got marked
- The system is inconsistent for $x_4 = 0$

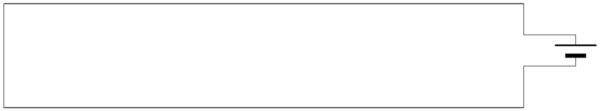
Circuit Lattice (Basic Construction)

- Circuit Lattice is a combination of switches and wires
- Two types of switches:

- 1-Switch controls vertical circuit by the horizontal
- 2-Switch controls horizontal circuit by the vertical

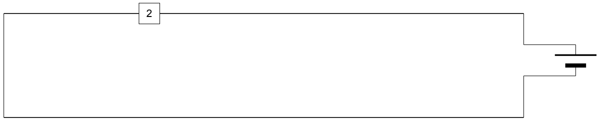
Constructing the Lattice

Local solution $\Leftrightarrow$ Horizontal circuit
Local solution wrong $\Leftrightarrow$ Potential in the circuit



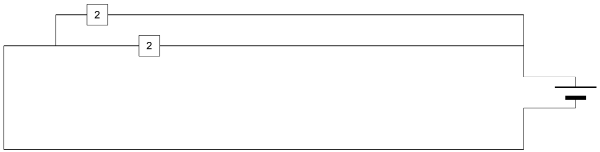
Constructing the Lattice

2-switch controls the circuit



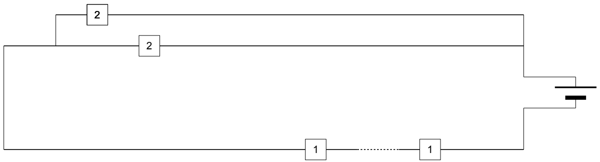
Constructing the Lattice

Several 2-switches may control the circuit



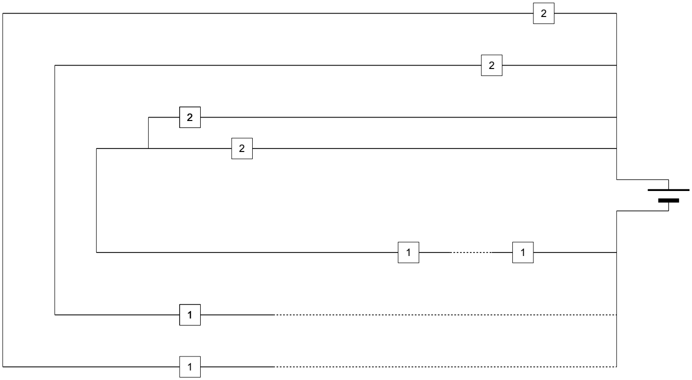
Constructing the Lattice

1-switches control some vertical circuits



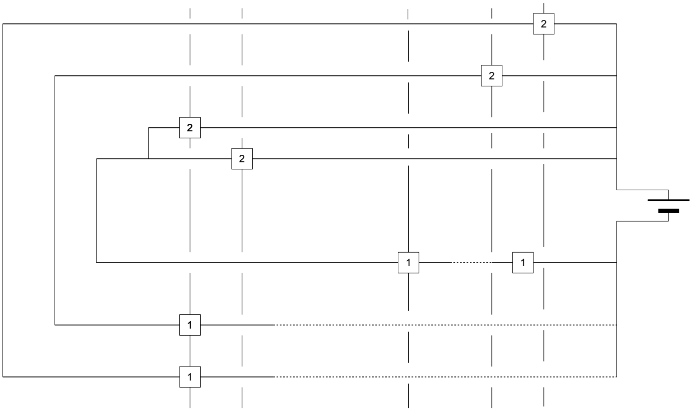
Constructing the Lattice

Many horizontal circuits



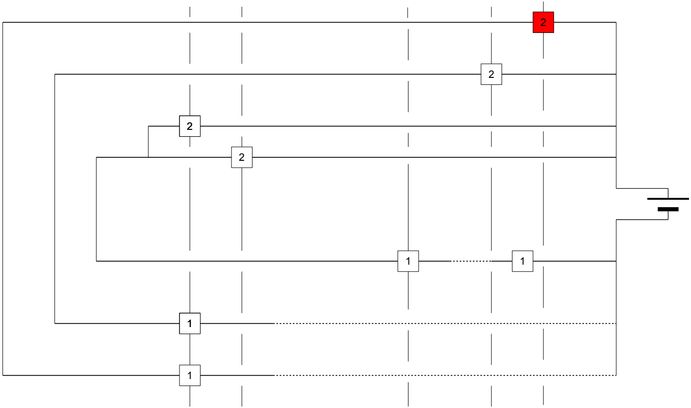
Constructing the Lattice

Tuple $\Leftrightarrow$ Vertical circuit



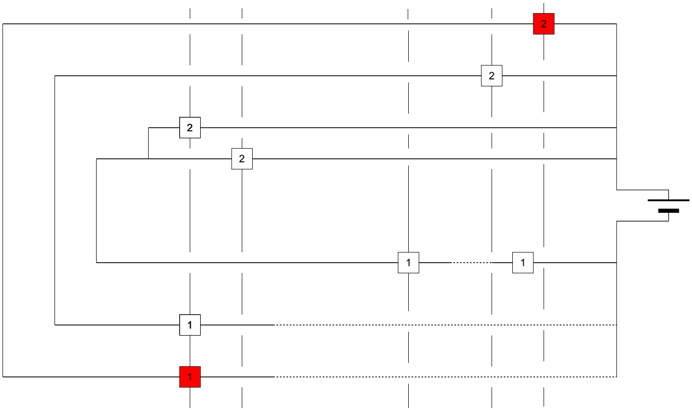
How It Works

Inducing potential in some circuits



How It Works

Expands potential to new circuits



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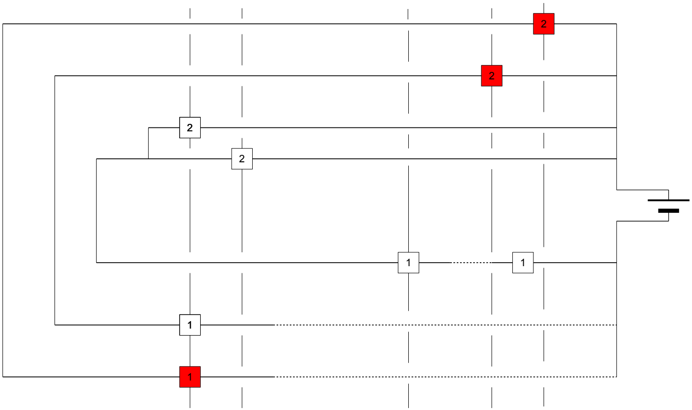

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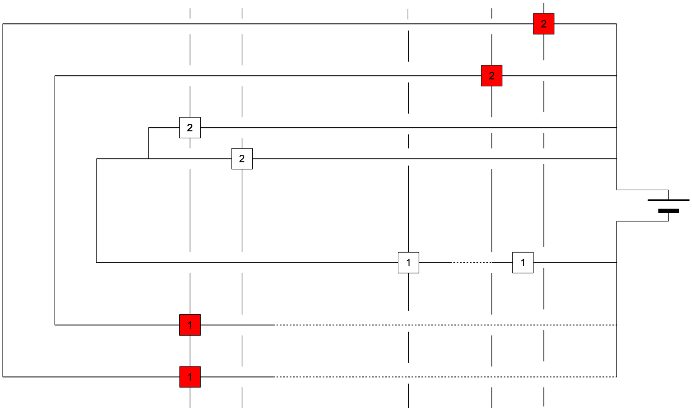

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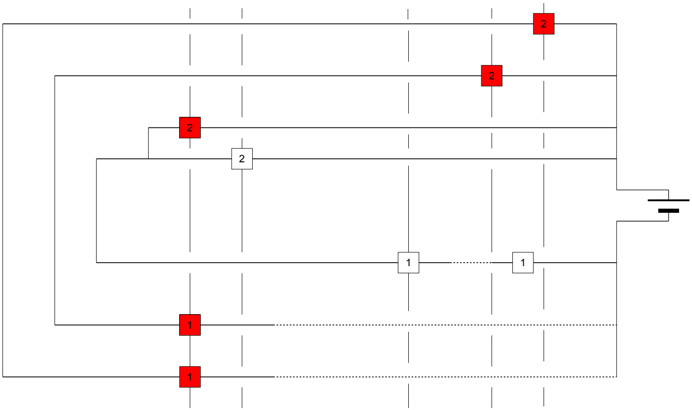

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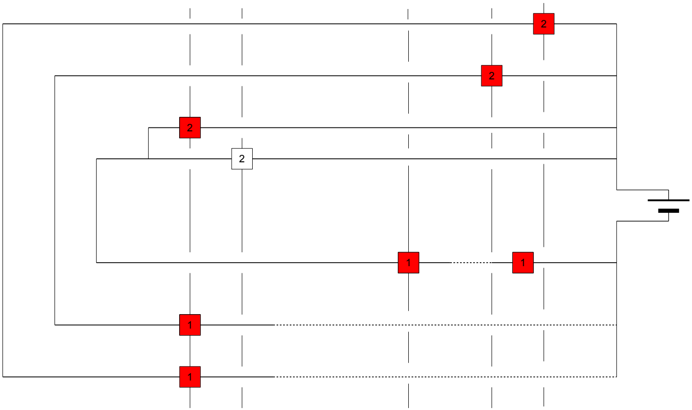
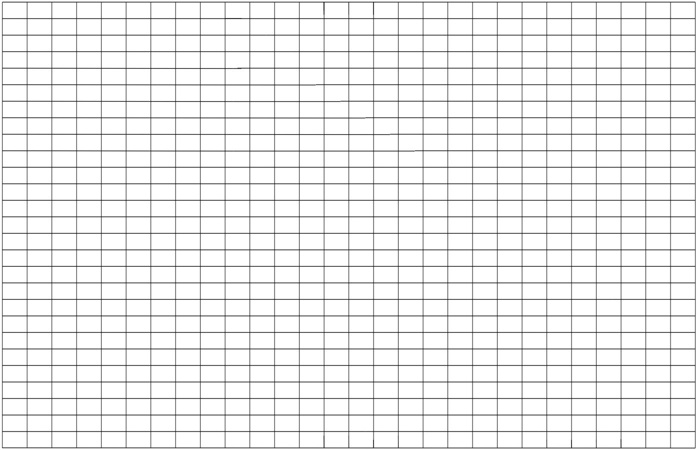


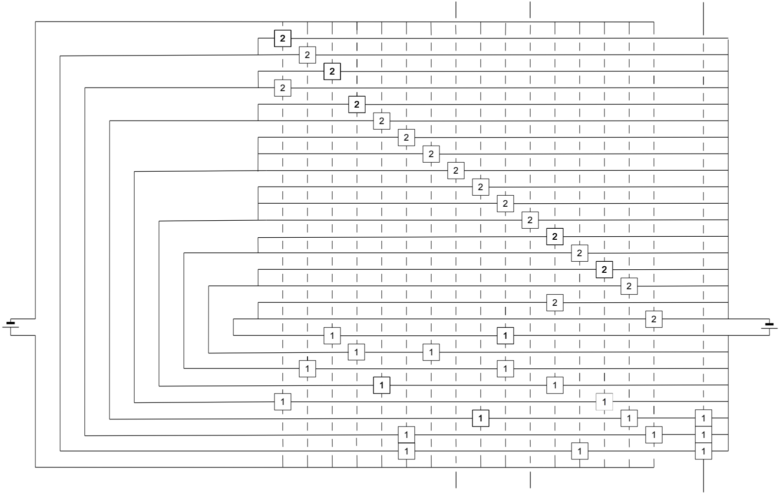
Introduce the guess

- Generally, no voltage in initial circuit lattice
- Assume E_i depends on x_j
- $a_1, \ldots, a_2$ solutions to E_i, where $x_j = 0$
- Add 2-Switch to each $a_1, \ldots, a_2$, connect them
- Guessing $x_j = 0$ is inducing voltage in new circuit
- Similarly, guessing $x_j = 1$
- s-variable guess - $2s$ new vertical circuits

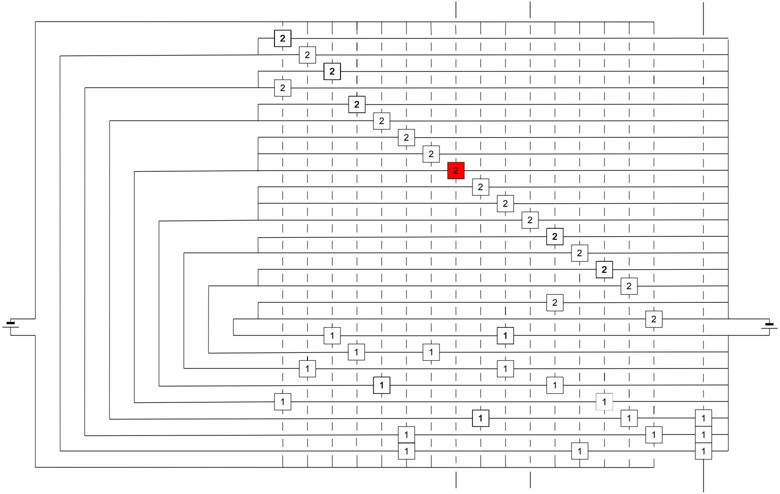
Grid Lattice



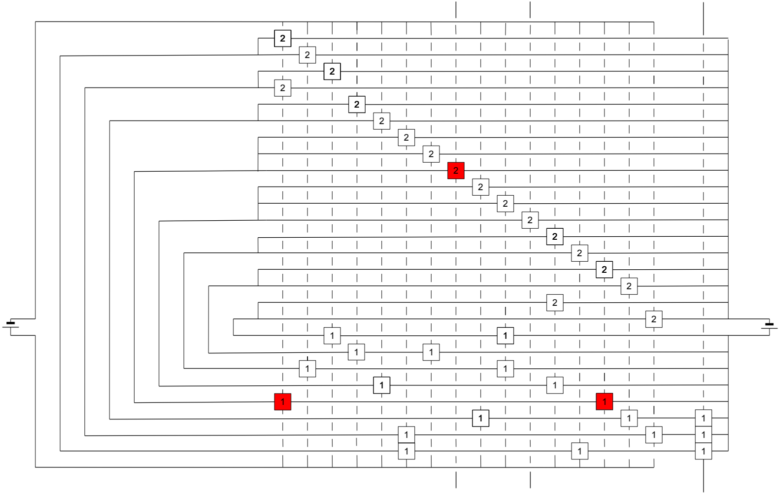
Exemplary Circuit Lattice



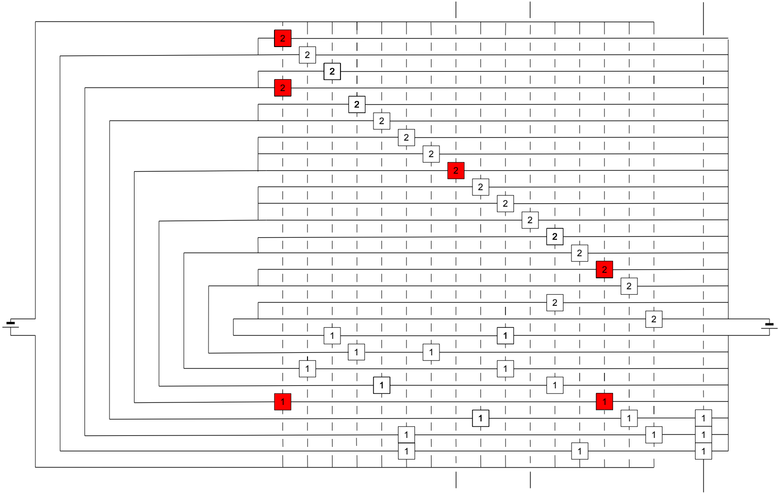
Exemplary Circuit Lattice. Introduce guess $x_4 = 0$



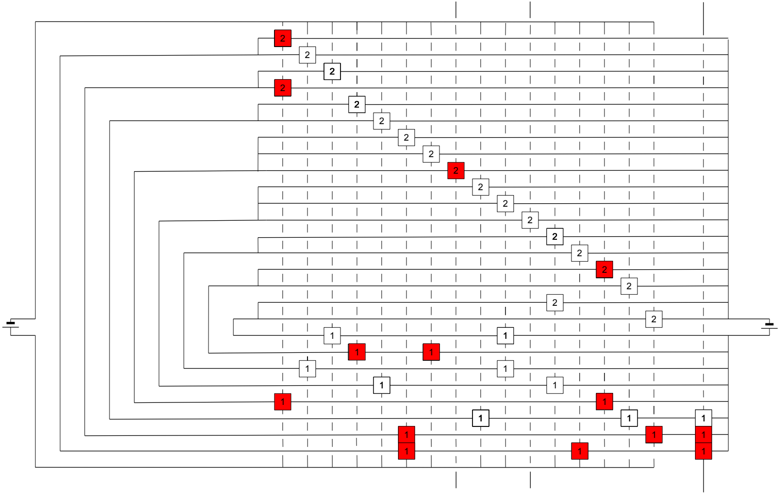
Exemplary Circuit Lattice



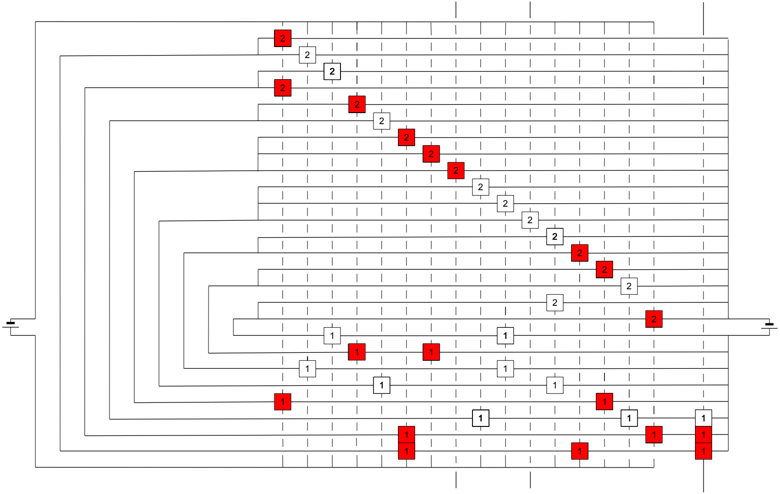
Exemplary Circuit Lattice



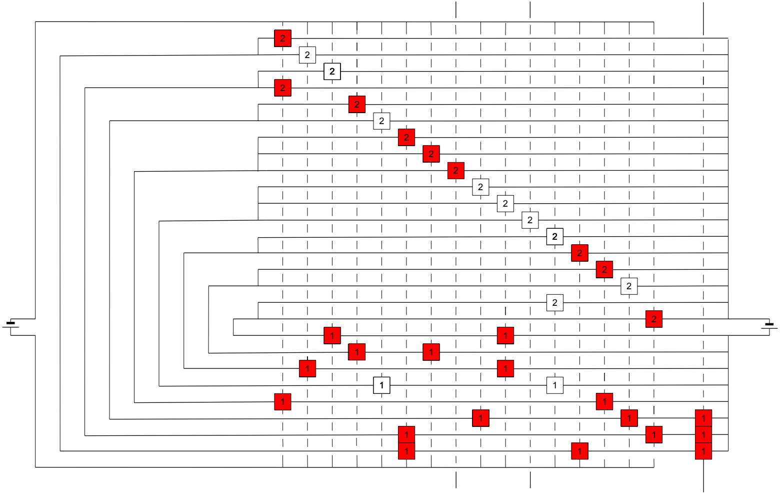
Exemplary Circuit Lattice



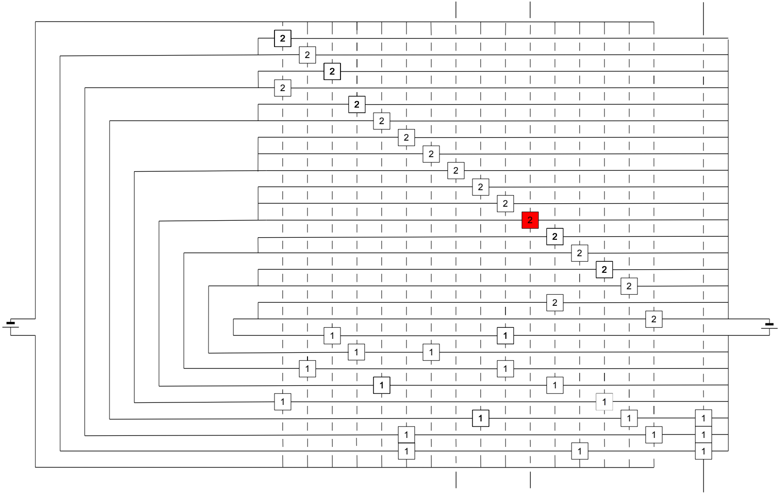
Exemplary Circuit Lattice



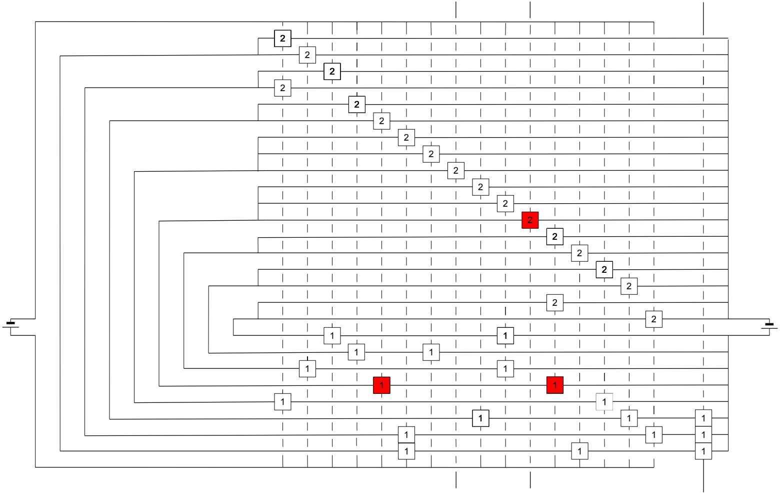
Exemplary Circuit Lattice



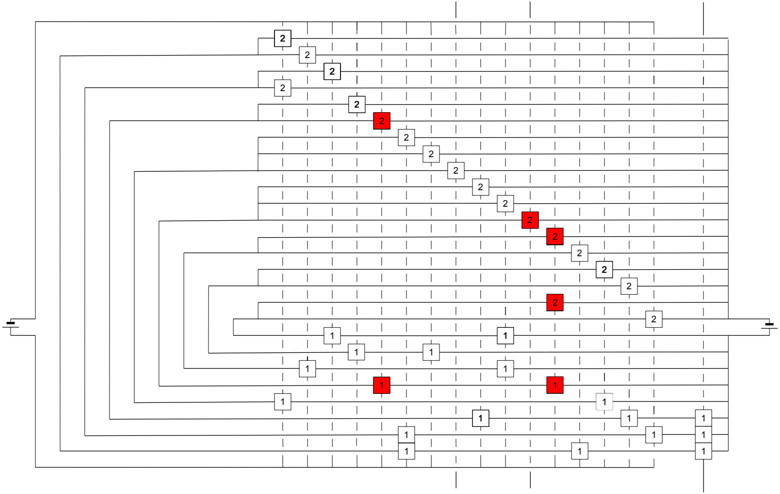
Exemplary Circuit Lattice. Introduce guess $x_4 = 1$



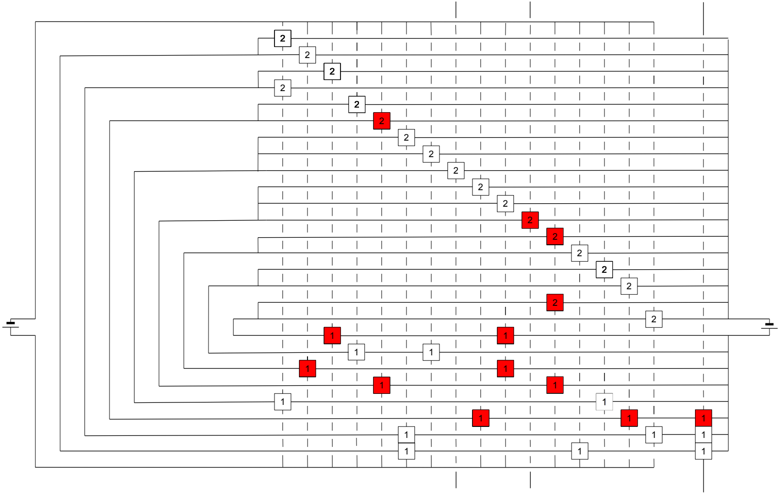
Exemplary Circuit Lattice



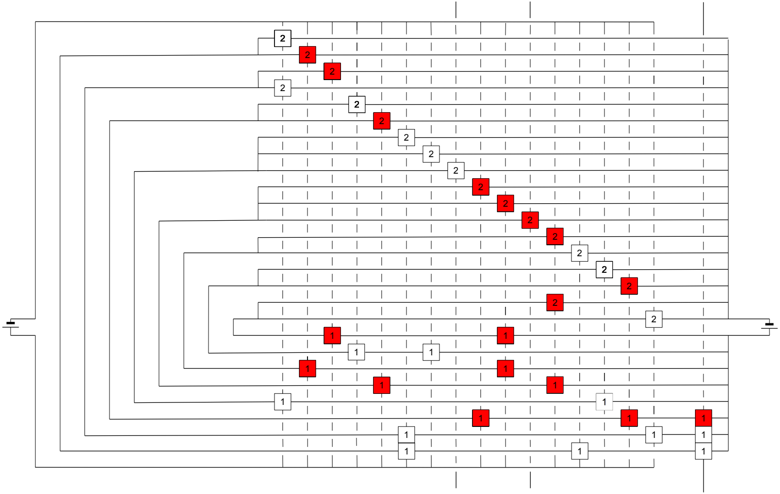
Exemplary Circuit Lattice



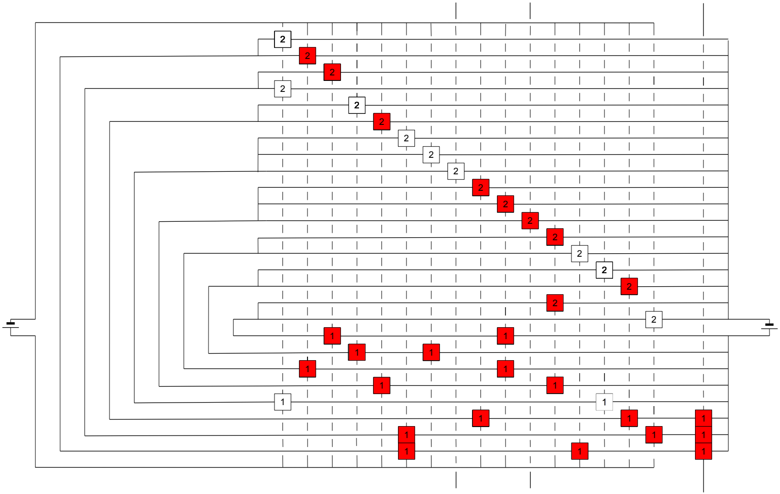
Exemplary Circuit Lattice



Exemplary Circuit Lattice



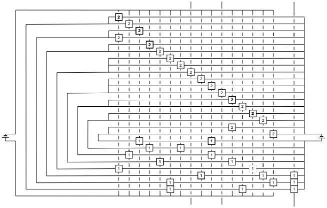
Exemplary Circuit Lattice



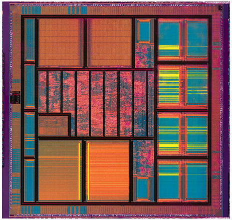
TripleDES system parameters

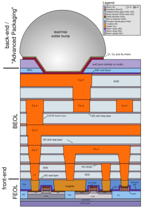
- 1712 variables, 384 equations
- 3929 maximal edges
- 71320 tuples
- 1.1×10^9 switches
- $480 = 2 \times 128 + 2 \times 112$ input contacts
- The device doesn't require synchronization

Circuit Lattice, as in WCC'09 topologically. Not much wiring intersection now. Implementable with two layers on a crystal.



Common Integrated Circuit, about 10 semiconductor layers





Implement on Modern Semiconductor Crystals for brute force?

- Transistor works as a switch
- 1.7×10^9 transistors on Dual-Core Itanium2 processor
- Circuit Lattice speed $\leq 2\times$(number of rounds) transistor turns
- $2 \times 48 + 2$ turns for TripleDES
- One transistor turn, say 100GHz(1000GHz reported)
- 1GHz key-rejecting rate when using for brute force
- Reported(2006) 0.13GHz per chip with implementing encryption

Conclusions

- WCC'09 design was improved
- Equation solving is shown as voltage expansion through a lattice of switches
- Our approach seems more flexible than implementing encryption as enables handling any equation system representing cipher
- Applications to DES, TripleDES are discussed