On a class of left MQQs with degree invariant to parastrophy

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Introduction - Left Quasigroups

(Q, q) - a groupoid, a - fixed element of Q.
(Q, q) is called **left quasigroup** if the mapping $L_a : Q \to Q$

$$L_a x = q(a, x),$$

is a bijection for every $a \in Q$.

If Q is finite, $|Q|$ - **order** of (Q, q).

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Introduction - Left Quasigroups

Parastrophe operation "$q_\backslash$": $q_\backslash(x,y)=z \Leftrightarrow q(x,z)=y$, defines a left quasigroup $(Q,q_\backslash)$.

Important identities:

$$q(x,q_\backslash(x,y))=y, \quad q_\backslash(x,q(x,y))=y,$$

(Q,q_1) and (Q,q_2) are **isotopic**,
if there exist bijections $\alpha,\beta,\gamma: Q\to Q$, s.t.

$$(\forall a,b\in Q) \quad \gamma(q_1(a,b))=q_2(\alpha(a),\beta(b)).$$

The isotopy is (α,β,γ).

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Introduction - Left Multivariate Quasigroups

Every left quasigroup (Q, q) of order 2^n:

$$\begin{aligned} & q(x,y) = z \Longleftrightarrow \\ & q(x_1,\ldots,x_n,y_1,\ldots,y_n) = \\ = \quad & (q^{(1)}(x_1,\ldots,x_n,y_1,\ldots,y_n),\ldots,q^{(n)}(x_1,\ldots,x_n,y_1,\ldots,y_n)). \end{aligned}$$

Each of the $q^{(s)}$ has a unique ANF form over $GF(2)$.

$$q^{(s)}(x_1,...,x_n,y_1,...,y_n) = \bigoplus_{\substack{j=(j_1,...,j_n)\in\mathbb{Z}_2^n \\ k=(k_1,...,k_n)\in\mathbb{Z}_2^n}} a_{jk} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} y_1^{k_1} \cdots y_n^{k_n},$$

where $a_{jk} \in \mathbb{Z}_2$, x^0 is an empty string and $x^1 = x$.

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Introduction - Left MQQs

If all $q^{(s)}$ are of degree 2 -

Left Multivariate Quadratic Quasigroups (MQQ)

- Suitable for symbolic computation
- Can be used in Multivariate Public Key cryptosystems

- Algorithms for construction
  - Gligoroski et al. (2008) - introduction of MQQ
  - Christov (2009) - characterization and algorithm for quadratic loops
  - Samardjiska et al. (2010) - characterization and algorithms for T-Multivariate Quasigroups and permutations of any degree
  - Chen et al. (2010)- algorithm for bilinear quadratic quasigroups

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Multivariate Public Key cryptosystems using MQQs

Typical scenario:

- Encryption: the (left) quasigroup q
- Decryption: the parastrophe $q_\backslash$:

$$q_\backslash(x, y) = z \Leftrightarrow q(x, z) = y.$$

Important:

- In the decryption process: The parastrophes are not in their ANF form
- Why? In general, time and space consuming!
  - Time: Can be difficult to find the ANF of the parastrophe
  - Space: Can have any degree and **exponentially** many terms

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So, is this a problem?

- MQQ-sig: bilinear quasigroups of order 2^8
- MQQ-enc: left quasigroups of order 2^8

Crucial: The decryption can be made by

- Using lookup tables - **the Caley table of the quasigroup (i.e. the parastrophe)**
- Solving small systems of multivariate quadratic equations

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But... what about other types of public key encryption like

Identity based encryption (IBE)?

Why IBE?

1. **The possibilities of IBE are enormous:**
   - No need for public key certificates - **public key = identity of its owner**
   - Revocation of public keys
   - Delegation of decryption keys
   - Generalization to more powerful HIBE, ABE, Functional encryption

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Can we use quasigroups to create a Multivariate IBE scheme?

Why is that important?

2. So far only Boneh-Franklin and Boneh-Boyen schemes are practical!
   - based on computational and decisional bilinear Diffie-Hellman problem

3. **A multivariate IBE has not been proposed so far!**

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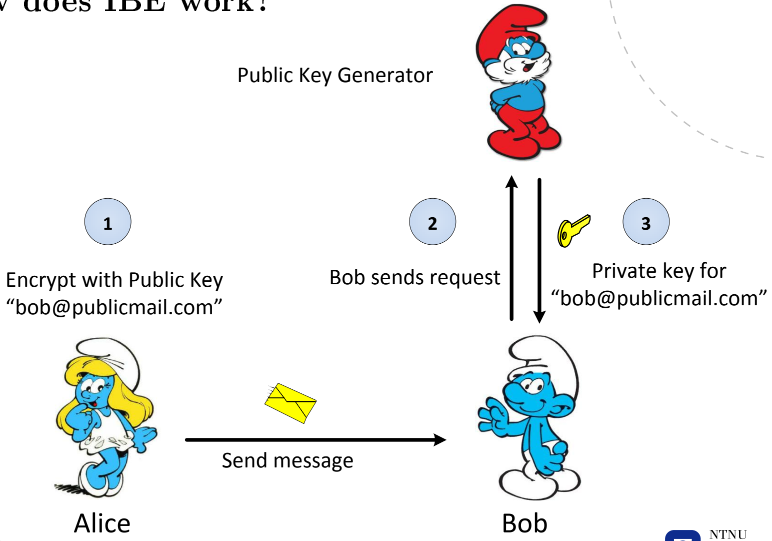
How does IBE work?



Public Key Generator

1

Encrypt with Public Key
“bob@publicmail.com”

2

Bob sends request

3

Private key for
“bob@publicmail.com”

Send message

Alice

Bob

Motivation

- The private key s (for decryption) for user ID should have **explicit multivariate form!**
- The quasigroups used can be **of orders as big as** 2^{64}, 2^{128}, 2^{256}!

Natural first step:

To find a class of left MQQs such that:

- The left parastrophe can be easily represented
- The left parastrophe is also a left MQQ, i.e. is of degree 2

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Construction of left quasigroups (SS 2010)

$x_1, \ldots, x_n, y_1, \ldots, y_n$ Boolean variables, $w > 1$.
$\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D}$ - nonsingular Bool. matrices, $\mathbf{c}$, $\mathbf{c_1}$, $\mathbf{c_2}$, $\mathbf{c_3}$, - Bool. vectors.
$\mathbf{A}$ and $\mathbf{B}$ - nonsingular upper triangular matrices of random affine Boolean expressions, such that:

- $\forall i = 1, \ldots, n$, $f_{ii} = 1$ and $g_{ii} = 1$, and
- $\forall i, j$, $i < j \leq n$, f_{ij} and g_{ij} depend only on $x_1, \ldots, x_n$, $y_{i+1}, \ldots, y_n$.

Then

$$\begin{aligned} &q(x_1, \ldots, x_n, y_1, \ldots, y_n) = \mathbf{A} \cdot (x_1, \ldots, x_n) + \mathbf{B} \cdot (y_1, \ldots, y_n) + \mathbf{c} \\ &q_1(x_1, \ldots, x_n, y_1, \ldots, y_n) = \\ &\qquad = \mathbf{D}(q(\mathbf{D}_1(x_1, \ldots, x_n) + \mathbf{c}_1, \mathbf{D}_2(y_1, \ldots, y_n) + \mathbf{c}_2)) + \mathbf{c}_3 \end{aligned}$$

define left MQQs (Q, q) and (Q, q_1) of order 2^n, $Q = \mathbb{Z}_2^n$.

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The first modification

$$q(\mathbf{x},\mathbf{y}) = \mathbf{A}(\mathbf{x},\mathbf{y})\cdot\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{y})\cdot\mathbf{y} + \mathbf{c} \qquad (SS2010)$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$q(\mathbf{x},\mathbf{y}) = \mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\cdot\mathbf{y} + \mathbf{c}$$

$\mathbf{A}(\mathbf{x})$ - vector of random quadratic Boolean expressions
$\mathbf{B}(\mathbf{x})$ - nonsingular upper triangular matrix, such that:

- $\forall i = 1,\ldots,n$, $b_{ii}(\mathbf{x}) = 1$, and
- $\forall i, j$, $i < j \le n$, $b_{ij}(\mathbf{x})$ random affine Boolean expressions of $x_1,\ldots,x_n$.

The parastrophe is

$$q_{\backslash}(\mathbf{x},\mathbf{y}) = \mathbf{B}^{-1}(\mathbf{x})\cdot\mathbf{y} + \mathbf{B}^{-1}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}) + \mathbf{B}^{-1}(\mathbf{x})\cdot\mathbf{c}$$

When is $q_\backslash(\mathbf{x}, \mathbf{y})$ quadratic?

$$q_\backslash(\mathbf{x}, \mathbf{y}) = \mathbf{B}^{-1}(\mathbf{x}) \cdot \mathbf{y} + \mathbf{B}^{-1}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}) + \mathbf{B}^{-1}(\mathbf{x}) \cdot \mathbf{c}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

Iff

- $\mathbf{B}^{-1}(\mathbf{x})$ has elements - affine expressions, and
- $deg(\mathbf{B}^{-1}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x})) = 2.$

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When are the elements of $\mathbf{B}^{-1}(\mathbf{x})$ affine expressions?

Iff

$\forall i, j,\ i \leq j \leq n$

$$\sum_{k=i}^{j} (\mathbf{x}^T B_{ik} \mathsf{B}_{kj}^T \mathbf{x} + \mathbf{x}^T B_{ik} \beta_{kj} + b_{ik} \mathsf{B}_{kj}^T \mathbf{x} + b_{ik} \beta_{kj}) = 0$$

where

- $\mathbf{B}(\mathbf{x}) : b_{ij}(\mathbf{x}) = \mathbf{x}^T \cdot B_{ij} + b_{ij}$, and B_{ij}, $\mathbf{x}$ column vectors,
- $\mathbf{B}^{-1}(\mathbf{x}) : \beta_{ij}(\mathbf{x}) = \mathbf{x}^T \cdot \mathsf{B}_{ij} + \beta_{ij}$, and B_{ij} column vector,

Construction

- expanded form, and
- backtracking algorithm,

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Construction

- expanded form, and
- backtracking algorithm,

Sufficient conditions

If $\forall i, j,\ i \leq j \leq n$

1. $\displaystyle\sum_{k=i}^{j} B_{ik}\mathsf{B}_{kj}^{T} = \mathbf{0}$
2. $\displaystyle\sum_{k=i}^{j} B_{ik}\beta_{kj} + b_{ik}\mathsf{B}_{kj}^{T} = \mathbf{0}$
3. $\displaystyle\sum_{k=i}^{j} b_{ik}\beta_{kj} = \mathbf{0}$

then the elements of $\mathbf{B}^{-1}(\mathbf{x})$ are affine expressions.

Still not good enough for construction!

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Lemma

If $\forall i, j,\ i \leq j \leq n$

1. $$\beta_{ij} = b_{ij} + \sum_{m=1}^{j-i} \sum_{i<r_1<\cdots<r_m<j} b_{ir_1} b_{r_1 r_2} \ldots b_{r_m j}$$
2. $$\mathsf{B}_{ij} = B_{ij} + \sum_{m=1}^{j-i} \sum_{\substack{i < r_1 < \cdots < r_m < j \\ t \in \{1,\ldots,m\}}} b_{ir_1} \ldots B_{r_t r_{t+1}} \ldots b_{r_m j}$$
3. $$\sum_{m=1}^{j-i} \sum_{i<r_1<\cdots<r_m<j} B_{ir_1} b_{r_1 r_2} \ldots b_{r_{m-1} r_m} B^T_{r_m j} = \mathbf{0}$$

then the elements of $\mathbf{B}^{-1}(\mathbf{x})$ are affine expressions.

Theorem for construction

Let $q(\mathbf{x},\mathbf{y}) = \mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x}) \cdot \mathbf{y} + \mathbf{c}$ where

- $\mathbf{B}(\mathbf{x}) : b_{ij}(\mathbf{x}) = \mathbf{x}^T \cdot B_{ij} + b_{ij}$, where B_{ij}, $\mathbf{x}$ column vectors,
- $\mathbf{A}(\mathbf{x}) : a_i(\mathbf{x})$ Boolean expressions.

If

- $B_{2k_1+1,2k_2} \neq \mathbf{0}$, $B_{2k_1+1,2k_2+1} = \mathbf{0}$, $B_{2k_1,2k_2+1} = \mathbf{0}$, $B_{2k_1,2k_2} = \mathbf{0}$,
- $b_{2k_1+1,2k_2} \neq 0$, $b_{2k_1+1,2k_2+1} \neq 0$, $b_{2k_1,2k_2} \neq 0$, $b_{2k_1,2k_2+1} = 0$,
- $a_{2k}(\mathbf{x})$ is affine and $a_{2k+1}(\mathbf{x})$ is quadratic,

Then q is a left MQQ with degree invariant to the parastrophe $\backslash$.

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Example

Let $\mathbf{A}(\mathbf{x})$ be a vector of dimension 4 and let $\mathbf{B}(\mathbf{x})$ be 4×4 matrix given by

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 1 + x_3 + x_1x_3 + x_2x_3 + x_4 \\ 1 + x_4 \\ 1 + x_2 + x_4 + x_3x_4 \\ 1 + x_1 + x_4 \end{bmatrix},$$

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 1 & x_1 + x_2 + x_3 & 1 & 1 + x_1 + x_3 + x_4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & x_1 + x_2 + x_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and let

$$q(\mathbf{x}, \mathbf{y}) = \mathbf{D}(\mathbf{A}(\mathbf{D_1} \cdot \mathbf{x} + \mathbf{c_1}) + \mathbf{B}(\mathbf{D_1} \cdot \mathbf{x} + \mathbf{c_1}) \cdot \mathbf{D_2} \cdot \mathbf{y} + \mathbf{c_2} + \mathbf{c}) + \mathbf{c_3}.$$

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Example

$$q(x_1,\dots,x_4,y_1,\dots,y_4) =$$

$$= \begin{bmatrix} 1+x_1+x_2+x_1x_2+x_3+x_1x_3+x_2x_3+x_4+x_1x_4+x_2x_4+y_1+ \\ +x_1y_1+x_4y_1+y_2+x_1y_2+x_2y_2+x_4y_2+x_1y_3+x_4y_3+x_2y_4 \\ 1+x_1x_2+x_3+x_1x_3+x_1x_4+x_2x_4+x_3x_4+y_1+x_1y_1+ \\ +x_4y_1+x_4y_2+x_1y_3+x_4y_3+x_1y_4 \\ x_1x_2+x_1x_3+x_1x_4+x_2x_4+x_3x_4+x_1y_1+x_4y_1+y_2+ \\ +x_4y_2+y_3+x_1y_3+x_4y_3+x_1y_4 \\ 1+x_1+x_3+x_2x_3+x_4+x_3x_4+x_1y_2+x_2y_2+y_4+x_1y_4+x_2y_4 \end{bmatrix}$$

is a Left MQQ of order 2^4.

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Example

$$q_{\backslash}(x_1,\dots,x_4,y_1,\dots,y_4) =$$

$$= \begin{bmatrix} x_2 + x_1x_3 + x_1x_4 + x_2x_4 + y_1 + x_1y_1 + y_2 + x_4y_2 + y_3 + x_4y_4 \\ x_1 + x_1x_2 + x_3 + x_4 + x_1x_4 + x_2x_4 + x_3x_4 + y_1 + x_2y_1 + x_1y_2 + \\ + x_2y_2 + x_4y_2 + y_4 + x_1y_4 + x_2y_4 + x_4y_4 \\ 1 + x_1 + x_1x_2 + x_1x_3 + x_4 + x_3x_4 + y_1 + x_1y_1 + x_2y_1 + x_1y_2 + \\ + x_2y_2 + y_3 + x_1y_4 + x_2y_4 \\ 1 + x_1 + x_2 + x_1x_2 + x_3 + x_4 + x_1x_4 + x_2x_4 + x_3x_4 + x_2y_1 + x_1y_2 + \\ + x_2y_2 + x_4y_2 + y_4 + x_1y_4 + x_2y_4 + x_4y_4 \end{bmatrix}$$

is again a Left MQQ of order 2^4.

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Thank you for your attention!

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